Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(99)00016-7
J. Appl. Maths Mechs, Vol. 63, No. 1, pp. 109–116, 1999

0021-8928/99/$—see front matter

THE CONTACT OF A SOLID WITH AN ELASTIC HALF-SPACE THROUGH A THIN COATING†

Ye. V. KOVALENKO

Moscow

(*Received* 16 *May* 1997)

More accurate equations of the deformation of thin plates, which are more convenient for solving contact problems for bodies with coatings and containing, as a special case, the equations of all known applied theories, are derived by an asymptotic analysis of the first fundamental problem of the theory of elasticity. The equations of the deformation of thin-walled elastic bodies are classified, their qualitative correspondence to the equations of the theory of elasticity is clarified, and the forms of the features that arise along the shift lines of the boundary conditions in the corresponding contact problems are established. A criterion for selecting approximate models to describe the properties of the coatings depending on the geometrical and mechanical characteristics of the coating and the substrate and also on their degree of adhesion is given.

The problem of refining and also of deriving new equations, suitable for solving mixed problems, is due to the fact that classical applied theories for describing practical phenomena are inadequate [1]. However, this situation does not mean that classical theories must be rejected. Despite the fact that there are a number of problems [2] where they do not work, there are problems that can only be solved using them [1]. Hence, the application of each theory of the deformation of thin-walled elastic components when solving any problem must be justified from the point of view of the agreement between the final result and some standard solution, obtained, for example, using the equations of the theory of elasticity.

1. We will first obtain the equations which describe the stress–strain state of thin plates. To do this we will study the problem of the equilibrium of an infinite elastic isotropic layer (see Fig. 1) (G is the shear modulus and ν is Poisson's ratio), which is in a state of three-dimensional non-axisymmetrical deformation under the action of normal and shear loads, applied to its faces. As regards the function $\sigma_m(x,y)$ and $\tau_{mn}(x,y)$ $(m,n = 1,2)$, we will assume that either they are piecewise-continuous, absolutely integrable and bounded over the whole plane (x,y), or are functions of limited variation.

The exact solution of this problem was constructed in [3] using a two-dimensional Fourier integral transformation with respect to the variables x and y and has the following structure ($\Gamma_1 = (-A, A)$, $\Gamma_2 = (-B, B)$)

$$\{\sigma_m, \tau_{mn}\} = \frac{1}{2\pi} \int_{\Gamma_1} \int_{\Gamma_2} \{\Sigma_m, T_{mn}\} e^{-i(\alpha x+\beta y)} d\alpha d\beta \tag{1.1}$$

$$\{u, \upsilon, w\} = \frac{1}{2\pi} \int_{\Gamma_1} \int_{\Gamma_2} \{U, V, W\} e^{-i(\alpha x+\beta y)} d\alpha d\beta \tag{1.2}$$

$$\begin{gathered} D_1(\alpha h, \beta h) U = P_1(\alpha h, \beta h, \alpha z, \beta z, \Sigma^{\pm}, T_m^{\pm}) \\ D_1(\alpha h, \beta h) V = P_2(\alpha h, \beta h, \alpha z, \beta z, \Sigma^{\pm}, T_m^{\pm}) \\ D_2(\alpha h, \beta h) W = P_3(\alpha h, \beta h, \alpha z, \beta z, \Sigma^{\pm}, T_m^{\pm}) \\ \Sigma^{\pm} = \Sigma_1 \pm \Sigma_2, \quad T_m^{\pm} = T_{1m} \pm T_{2m} \ (m = 1, 2) \end{gathered} \tag{1.3}$$

where u, υ, and w are the components of the displacement vector of points of the elastic layer.

Here it is essential that the limits of integration in the quadratures (1.1) should be finite [4] and satisfy the inequalities

†*Prikl. Mat. Mekh.* Vol. 63, No. 1, pp. 119–127, 1999.

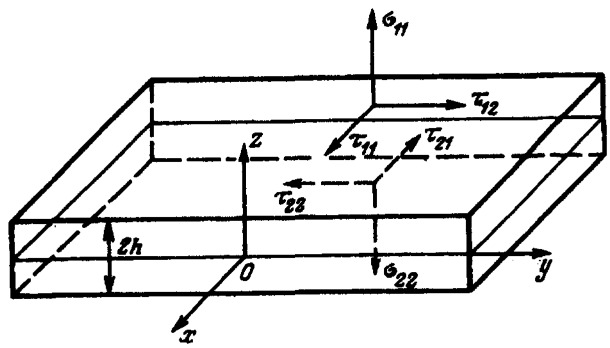

Fig. 1.

$$Ah \ll 1, Bh \ll 1 \tag{1.4}$$

These relations serve as one of the "conditions of applicability of the theory of thin plates" and indicate that the external stresses are distributed smoothly over the plate surfaces $z = \pm h$.

We will introduce a dimensionless parameter, determined by the thickness $2h$ and the geometry of the region Ω of active loading of the layer, i.e. that region where the surface loads $\sigma_m(x, y)$ and $\tau_{mn}(x, y)$ $(m, n = 1, 2)$ constitute, for example, no less than 5% of the maximum values. Thus, when the region Ω is simply connected and bounded and its contour has a continuous curvature, we can assume [5]

$$\lambda = h[\min(l, \rho_{\min})]^{-1}$$

$$l = \tfrac{1}{2}\max\sqrt{(x-\xi)^2 + (y-\eta)^2}\ (x, y) \in \Omega, (\xi, \eta) \in \Omega$$

where $\rho_{\min}$ is the minimum radius of curvature of the boundary $\partial\Omega$.

A second consequence of the finiteness of the limits of integration in (1.1), in agreement with conditions (1.4), is the fact that the theory of thin plates only holds [4] when $\lambda \ll 1$.

We will assume that $z = \pm h$ in (1.3) asymptotically, apart from terms of the order of $\lambda^3(\lambda \to 0)$, we will simplify the first two expressions and retain terms $O(\lambda^4)$ in the last equation. Then, returning to the originals using formulae (1.2) [3], we can write (correcting the errors on the right-hand sides of the similar equations given in [3, 6, 7])

$$\tfrac{4}{3}\theta h^2 \Delta^2 u_\pm = A^\pm_{1,x}(\boldsymbol{\sigma}) + A^\pm_{2,xx}(\boldsymbol{\tau}_1) + A^\pm_{3,yy}(\boldsymbol{\tau}_1) + A^\pm_{4,xy}(\boldsymbol{\tau}_2) \tag{1.5}$$

$$\tfrac{4}{3}\theta h^2 \Delta^2 v_\pm = A^\pm_{1,y}(\boldsymbol{\sigma}) + A^\pm_{2,yy}(\boldsymbol{\tau}_2) + A^\pm_{3,xx}(\boldsymbol{\tau}_2) + A^\pm_{4,xy}(\boldsymbol{\tau}_1) \tag{1.6}$$

$$\tfrac{4}{3}\theta h^3 \Delta^2 w_\pm = B^\pm_1(\boldsymbol{\sigma}) + B^\pm_{2,x}(\boldsymbol{\tau}_1) + B^\pm_{2,y}(\boldsymbol{\tau}_2) \tag{1.7}$$

Here $\theta = G(1-\nu)^{-1}$ and Δ is the two-dimensional Laplace operator and we have put

$$\{u_\pm, v_\pm, w_\pm\} = \{u(x, y, \pm h), v(x, y, \pm h), w(x, y, \pm h)\}$$

$$\boldsymbol{\sigma} = \{\sigma^-, \sigma^+\},\quad \boldsymbol{\tau}_j = \{\tau_j^-, \tau_j^+\},\quad \mathbf{a} = \{a^-, a^+\}$$

$$\sigma^\pm = \sigma_1 \pm \sigma_2,\quad \tau_j^\pm = \tau_{1j} \pm \tau_{2j}\quad (j = 1, 2)$$

$$A_1^\pm(\mathbf{a}) = \mp a^- \pm \frac{2-7\nu}{15(1-\nu)} h^2 \Delta a^- - \frac{\nu}{1-\nu} h^2 \Delta a^+$$

$$A_2^\pm(\mathbf{a}) = \tfrac{1}{3} h(-a^- \pm 3a^+ + \tfrac{2}{3} h^2 \Delta a^- \pm \tfrac{2}{5} h^2 \Delta a^+)$$

$$A_3^\pm(\mathbf{a}) = \frac{2h}{3(1-\nu)}(-a^- + \frac{1}{3} h^2 \Delta a^- \pm h^2 \Delta a^+)$$

$$A_4^\pm(\mathbf{a}) = h\left[\frac{1+\nu}{3(1-\nu)} a^- \pm a^+ - \frac{2\nu}{9(1-\nu)} h^2 \Delta a^- \mp \frac{2(4+\nu)}{15(1-\nu)} h^2 \Delta a^+\right]$$

$$B_1^{\pm}(\mathbf{a}) = a^- - \frac{4}{5}h^2\Delta a^- + \frac{27}{175}h^4\Delta^2 a^- \pm \frac{1}{3}h^4\Delta^2 a^+$$

$$B_2^{\pm}(\mathbf{a}) = h\left[a^+ - \frac{2-7\nu}{15(1-\nu)}h^2\Delta a^+ \pm \frac{\nu}{3(1-\nu)}h^2\Delta a^-\right]$$

Relations (1.5)–(1.7) are more accurate differential equations of the deformation of thin plates (coatings) and enable one to take into account both their longitudinal and transverse deformations due to stretching and shear, and their deformation due to transverse bending and compression.

Note that if, when deriving the more accurate equations of the deformation of thin plates, in view of the fact that the parameter λ is small, we average expressions (1.3) over the thickness

$$\{\tilde U, \tilde V, \tilde W\} = \frac{1}{2h}\int_{-h}^{h}\{U, V, W\}dz$$

we arrive at the following equations

$$\tfrac{4}{3}\theta h\Delta^2 \tilde u = \tilde A_{1,x}(\sigma^+) + \tilde A_{2,xx}(\tau_1^-) + \tilde A_{3,yy}(\tau_1^-) + \tilde A_{4,xy}(\tau_2^-) \tag{1.8}$$

$$\tfrac{4}{3}\theta h\Delta^2 \tilde v = \tilde A_{1,y}(\sigma^+) + \tilde A_{2,yy}(\tau_2^-) + \tilde A_{3,xx}(\tau_2^-) + \tilde A_{4,xy}(\tau_1^-) \tag{1.9}$$

$$\tfrac{4}{3}\theta h^3\Delta^2 w = \tilde B_1(\sigma^-) + \tilde B_{2,x}(\tau_1^+) + \tilde B_{2,y}(\tau_2^+) \tag{1.10}$$

$$\tilde A_1(a^+) = -\frac{\nu h}{3(1-\nu)}\Delta a^+, \quad \tilde A_2(a^-) = -\frac{1}{3}a^- - \frac{\nu}{9(1-\nu)}h^2\Delta a^-$$

$$\tilde A_3(a^-) = -\frac{2}{3(1-\nu)}a^-, \quad \tilde A_4(a^-) = \frac{1+\nu}{3(1-\nu)}a^- - \frac{\nu}{9(1-\nu)}h^2\Delta a^-$$

$$\tilde B_1(a^-) = a^- - \frac{12-7\nu}{15(1-\nu)}h^2\Delta a^- + \frac{11}{525}h^4\Delta^2 a^-$$

$$\tilde B_2(a^+) = h(a^+ - \frac{2}{15}h^2\Delta a^+)$$

which, unlike (1.5)–(1.7), only take into account the longitudinal deformations due to extension–compression and bending.

The proposed models (1.5)–(1.10) differ from the equations obtained in [8] by the presence of higher-order terms in λ on the right-hand sides and in the form of the coefficients of λ. For example, in Eq. (1.10) the difference when $\nu = 0.3$ is 3% in the coefficients of λ^2 and 21% in the coefficients of λ^3.

We will consider in more detail special cases of the equations of the deformation of thin plates (1.5)–(1.10) derived above.

Neglecting terms of the order of λ^2 and higher on the right-hand side of (1.10), we obtain the Kirchhoff–Love model of a plate. Note that similar equations can be obtained from relations (1.7). If we only drop terms $O(\lambda^4)$ in (1.10), we obtain equations of the Reissner–Timoshenko type theory. More exact questions of this theory can be written using expressions (1.7).

On the left- and right-hand sides of relations (1.10) (or 1.7)) we will drop terms of the order of λ^2 and higher (i.e. we will neglect the bending stiffness of the plate), while on the right-hand sides of (1.8) and (1.9) (or (1.5) and (1.6)) terms $O(\lambda^3)$, and convert the last two expressions using the first. We obtain

$$\begin{aligned} 4Gh\Delta^2 \tilde u &= -\nu h\Delta\sigma^+_{,x} - (1-\nu)\tau^-_{1,xx} - 2\tau^-_{1,yy} + (1+\nu)\tau^-_{2,xy} \\ 4Gh\Delta^2 \tilde v &= -\nu h\Delta\sigma^+_{,y} - (1-\nu)\tau^-_{2,yy} - 2\tau^-_{2,xx} + (1+\nu)\tau^-_{1,xy} \\ \sigma^- &= -h(\tau^+_{1,x} + \tau^+_{2,y}) \end{aligned} \tag{1.11}$$

System (1.11) is the equations of the deformation of a Melan covering [6].

Suppose now that $\tau_{1j} = u_- = \upsilon_- = w_- = 0$ ($j = 1, 2$) in (1.5)–(1.7). We will neglect terms $O(\lambda^2)$ and higher in Eq. (1.7) for w_-. We obtain

$$\sigma^- = -h(\tau_{21,x} + \tau_{22,y}) \quad (1.12)$$

At the same time, expressions (1.5) and (1.6) for u_- and υ_- give

$$\nu h(1-\nu)^{-1}\Delta\sigma^+ = \tau_{21,x} + \tau_{22,y} \quad (1.13)$$

Introducing (1.13) into (1.7) for w_+, we obtain

$$w_+ = (1-2\nu)h[2G(1-\nu)]^{-1}\sigma^+ \quad (1.14)$$

Expressions (1.12) and (1.14) are the equations of a Fuss–Winkler foundation, describing the deformation of a coating due to transverse compression.

Moreover, the retention of terms of the order of λ^6 when deriving (1.5)–(1.7) enables us, instead of (1.12) and (1.14), to write the three-dimensional analogue of the equation of a Pasternak–Vlasov foundation [9] with several coefficients of the bed and more accurate equations of the shear deformations [7].

2. We will now consider contact problems of the indentation without friction with a force P and momenta M_x and M_y of a rigid punch into a composite foundation, which an elastic (G_2, γ_2) half-space, reinforced with a relatively thin elastic (G_1, γ_1) layer of thickness h. We will assume that the coating either lies freely on an elastic substrate (problem 1) or is rigidly attached to it (problem 2). The physical-mechanical properties of the half-space will be described by the equations the theory of elasticity (Lamé's equations), while the coatings will be described by Lamé's equations and the equations of the applied theories, presented in Section 1, respectively. Then, using an integral Fourier transformation with respect to x and y, the problems reduce to determining the contact pressure $q(x, y) = -\sigma_z(x, y, h)\theta_2^{-1}$ ($\theta_2 = G_2(1-g_2)^{-1}$) from an integral equation of the form

$$\mathbf{F}q = g\ ((x, y) \in \Omega) \quad (2.1)$$

$$\mathbf{F}q = \iint_\Omega q(\xi,\eta)k(\xi - x, \eta - y)d\xi d\eta$$

$$k(s,t) = \frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} K(h\gamma)e^{i(\alpha s+\beta t)}d\alpha d\beta,\ \gamma^2 = \alpha^2 + \beta^2 \quad (2.2)$$

In (2.1) $g(x,y) = \delta + \alpha_y x + \alpha_x y - f(x,y)$ is a function of the settlement, defined by the rigid displacement of the punch and the shape of its base $z = f(x, y)$, which occupies the region Ω.

The symbol of the kernel $K(h\gamma)$ in (2.2) can be represented as $K(h\gamma) = L(h\gamma)\gamma^{-1}$, where the following expressions are obtained for the function $L(u)$ ($u = h\gamma$) [10]

$$1)\ nL(u) = \frac{\operatorname{ch}2u - 1 + n(\operatorname{sh}2u + 2u)}{\operatorname{sh}2u + 2u + n(\operatorname{ch}2u - 1 - 2u^2)} \quad (2.3)$$

$$2)\ nL(u) = \frac{2n\operatorname{ch}2u + (2m+\theta)\operatorname{sh}2u + 2\theta u}{(2m+\theta)(\operatorname{ch}2u - 1) + 2n\operatorname{sh}2u + 2(1-\theta u^2)} \quad (2.4)$$

$$\theta = n^2 - m^2,\ m = 1-\mu,\ \mu = \varepsilon_1 - n\varepsilon_2,\ n = \theta_1\theta_2^{-1}$$

$$\theta_1 = G_1(1-\nu_1)^{-1},\ \varepsilon_j = (1-2\nu_j)[2(1-\nu_j)]^{-1} (j = 1,2)$$

when the stress–strain state of the coating is modelled by the equations of the theory of elasticity and

$$nL(u) = [n + R_1(u)][1 + R_2(u)]^{-1} \quad (2.5)$$

when the properties of the coating are described using the applied theories.

In particular, we have for the first problem

$$R_1(u) = \sum_{i=1}^{5} a_{1i}u^i,\ R_2(u) = \sum_{i=2}^{4} b_{1i}u^i \quad (2.6)$$

$$a_{11} = \frac{1}{2},\ a_{12} = \frac{1}{5}n,\ a_{13} = \frac{1}{10},\ a_{14} = \frac{16}{525}n$$

$$a_{15}=\frac{27}{5600},\ b_{12}=\frac{1}{5},\ b_{13}=\frac{1}{6}n,\ b_{14}=\frac{1}{n}a_{14}$$

$$R_1(u)=a_{22}u^2+a_{24}u^4,\ R_2(u)=\sum_{i=2}^{4}b_{2i}u^i \tag{2.7}$$

$$a_{22}=\frac{n}{60}(17-10\varepsilon_1),\ a_{24}=\frac{11}{8400}n,\ b_{22}=\frac{1}{n}a_{22},\ b_{23}=b_{13},\ b_{24}=\frac{1}{n}a_{24}$$

$$R_1(u)=a_{32}u^2,\ R_2(u)=b_{32}u^2+b_{33}u^3 \tag{2.8}$$

$$a_{32}=a_{22},\ b_{32}=b_{22},\ b_{33}=b_{13}$$

$$R_1(u)=0,\ R_2(u)=b_{43}u^3,\ b_{43}=b_{13} \tag{2.9}$$

$$R_1(u)=a_{51}u,\ R_2(u)=0,\ a_{51}=a_{11} \tag{2.10}$$

in the case of Eqs (1.7) and (1.10), and theories of the Reissner–Timoshenko and Kirchhoff–Love type and of the Fuss–Winkler foundation, respectively.

For the second problem we obtain

$$R_1(u)=\sum_{i=1}^{5}c_{1i}u^i,\ R_2(u)=\sum_{i=1}^{4}d_{1i}u^i \tag{2.11}$$

$$c_{11}=2(n^2-\mu^2+\mu),\ c_{12}=\frac{28}{15}n,\ c_{13}=\frac{1}{5}c_{11}+\frac{1}{12}(7-6\mu)$$

$$c_{14}=\frac{781}{2100}n,\ c_{15}=-\frac{1}{4200}(94n^2+256\mu^2-186\mu-157)$$

$$d_{11}=2n,\ d_{12}=2\left(\frac{14}{15}-\mu\right),\ d_{13}=\frac{16}{15}n,\ d_{14}=\frac{1}{6}c_{11}+\frac{1}{525}(64-35\mu)$$

$$R_1(u)=\sum_{i=1}^{4}c_{2i}u^i,\ R_2(u)=\sum_{i=1}^{4}d_{2i}u^i \tag{2.12}$$

$$c_{21}=n[2n-\varepsilon_2(1-2\mu)],\ c_{22}=\frac{n}{20}(10\varepsilon_1-1),\ c_{23}=\frac{1}{n}a_{22}c_{21}$$

$$c_{24}=S_{21}(n),\ d_{21}=d_{11},\ d_{22}=n\varepsilon_2-\frac{1}{2}\varepsilon_1+\frac{9}{20},\ d_{23}=\frac{n}{15}(11-5\varepsilon_1),\ d_{24}=S_{22}(n)$$

$$R_1(u)=\sum_{i=1}^{4}c_{3i}u^i,\ R_2(u)=\sum_{i=1}^{4}d_{3i}u^i \tag{2.13}$$

$$c_{31}=c_{21},\ c_{32}=\frac{n}{30}(10\varepsilon_1+1),\ c_{33}=c_{23},\ c_{34}=-\frac{n}{120}(1-2\varepsilon_1)$$

$$d_{31}=d_{21},\ d_{32}=\frac{1}{15}(15n\varepsilon_2-10\varepsilon_1+8),\ d_{33}=d_{23},\ d_{34}=S_{32}(n)$$

$$R_1(u)=\sum_{i=1}^{4}c_{4i}u^i,\ R_2(u)=\sum_{i=1}^{4}d_{4i}u^i \tag{2.14}$$

$$c_{41}=c_{21},\ c_{42}=\frac{n}{4}(2\varepsilon_1-1),\ c_{43}=0,\ c_{44}=10nc_{34}$$

$$d_{41}=d_{21},\ d_{42}=\frac{1}{4}(4n\varepsilon_2-2\varepsilon_1+1),\ d_{43}=\frac{n}{6},\ d_{44}=\frac{n^2}{3}$$

$$R_1(u)=a_{51}u,\ R_2(u)=d_{51}u,\ c_{51}=c_{21},\ d_{51}=d_{21} \tag{2.15}$$

$$R_1(u) = c_{61}u, \quad R_2(u) = 0, \quad c_{61} = c_{11} \tag{2.16}$$

when the properties of the coating are specified by Eqs (1.5)–(1.7), (1.8)–(1.10) and ($1.11_{1,2}$) and the Reissner–Timoshenko type theory ($1.11_{1,2}$) and the Kirchhoff–Love type theory, and (1.11) and (1.12) and (1.14). Here $S_{ki}(n)$ are polynomials of degree j, the form of which is omitted for brevity.

To close the formulation of the problems in question on the indentation of a punch into a two-layer foundation for a fixed region Ω we must add the following statics conditions to integral equation (2.1), (2.2)

$$P = \theta_2 \iint_\Omega q(x,y)dxdy, \quad \{M_x, M_y\} = \theta_2 \iint_\Omega \{y,x\} q(x,y)dxdy \tag{2.17}$$

which serves to establish the relation between the force factors P, M_x and M_y and the geometrical factors δ, α_x, α_y.

We will now dwell in more detail on an asymptotic analysis of the problems, assuming that

$$n = O(\lambda^\kappa) \quad (\lambda \to 0) \tag{2.18}$$

where, when $\kappa > 0$, the stiffness of the elastic substrate is greater than the stiffness of the coating, and, conversely, when $\kappa < 0$, the stiffness of the half-space is less than the stiffness of the reinforcing layer. To do this we obtain the difference between the functions (2.3) and (2.4) and the corresponding expressions (2.5)–(2.16), we introduce their relation (2.18) and we establish the order of the result obtained as $\lambda \to 0$.

We will initially investigate the first problem and assume that $\kappa \geqslant 0$. Dropping terms $O(\lambda^5)$ and higher we see that formulae (2.3) and (2.5) and (2.6) are identical. When $0 \leqslant \kappa < 1$, apart from terms of the order of $\lambda^{3+2\kappa}$, and when $\kappa \geqslant 1$, apart from terms $O(\lambda^5)$, we establish the closeness of relations (2.3) and (2.5), (2.10). Note that when $\kappa \geqslant 3$ in practical calculations the first term in (2.5) and (2.10) are usually neglected with an accuracy of the order of $O(\lambda^3)$ [7] and we consider the coating as a thin Winkler layer, lying on a rigid foundation.

Suppose now that $-3 \geqslant \kappa < 0$. Then, omitting terms $O(\lambda^{5+\kappa})$ and higher, we obtain equality of the relations (2.3) and (2.5), (2.6). At the same time, for $-2 \geqslant \kappa < 0$, neglecting terms $O(\lambda^{1-\kappa})$, while for $-3 \geqslant \kappa < -2$, neglecting quantities of the order $(\lambda^{5+\kappa})$, we have agreement between the function (2.3) and one of the formulae (2.5) and (2.7)–(2.9). When $\kappa < -3$, apart from terms $O(\lambda^{-1-\kappa})$, we obtain equality between the function (2.3) and one of the functions of the form (2.5), (2.6)–(2.9).

We now consider the second problem and we will assume that the parameter $\kappa \geqslant 0$. Dropping terms of the order of λ^3 and higher we conclude that formulae (2.4) and (2.5), (2.11) agree with one another. Together with this, when $0 < \kappa < 2$, apart from terms of the order of $O(\lambda^{1+\kappa})$ and when $\kappa \geqslant 2$, apart from terms of the order of λ^3, we obtain equality between relations (2.4) and (2.5), (2.16). When $\kappa \geqslant 3$, as above, apart from terms up to $O(\lambda^3)$ in engineering calculations [11, 12] we assume the substrate to be rigid while the coating is simulated by a set of Winkler springs.

We will now assume that $\kappa < 0$. For this version we have agreement between formulae (2.4) and (2.5), (2.11) up to terms of the order of λ^3, and agreement between equality (2.4) and one of relations (2.5), (2.12)–(2.15), apart from terms of the order of $O(\lambda)$.

Hence, we can draw the following conclusions from the above asymptotic analysis.

1. equations (1.5)–(1.7) are applicable, with a fairly high degree of accuracy, over the whole range of variation of the main geometrical and physical parameters of the problems in question;

2. the deformation properties of a "soft" coating agree well with the Fuss–Winkler hypothesis;

3. in the case of a "rigid" coating and weak adhesion to the substrate, the accuracy of Eqs (1.10) and their modifications increases as the parameter n increases, agreeing with the accuracy of Eqs (1.7) when $n \approx O(\lambda^{-2})$; an increase in the physical parameter n from $n \approx O(\lambda^{-2})$ to $n \approx O(\lambda^{-3})$ leads to a small loss of accuracy in the equations of all the theories considered in Section 1; finally, a further increase in the relative stiffness of the coating n is accompanied by an increase in the accuracy of all the equations of the applied theories of the deformation of elastic plates;

4. for a "rigid" coating, adhering to the base, the use of Eqs (1.8)–(1.10) and their modifications leads to an accuracy given by the Melan covering model (1.11), and hence for this version it is best to use either Eqs (1.5)–(1.7) or relations (1.11).

3. We will study the structure of the solution of integral equation (2.1), (2.2), which depends very much on the behaviour of the symbol $K(u)$ of its kernel $k(s,t)$ on the real axis (below, in Eq. (2.2_2) we will assume [4] $\Gamma_1 = \Gamma_2 = (-\infty, \infty)$. For the problems considered in Section 2, $K(u)$ is a positive, even and continuous function (apart from the point $u = 0$) when $|u| < \infty$, and has the following asymptotic form

$$K(u) = D_m |u|^{-1} \{1 + O(|u|^{-\alpha} e^{-\beta|u|}) \ (|u| \to \infty), \quad K(u) \sim |u|^{-1} \ (u \to 0) \tag{3.1}$$

where D_m = const ($m = 0, 1, 2, 4$), while $\alpha = 0$ and $\beta = 2$ or $\alpha \geq 1$, $\beta = 0$.

We will now introduce necessary definitions and notation [13–15]. Suppose $C^k(\Omega)$ is the set of k-times continuously differentiable functions in the region Ω, $C^0(\Omega)$, C_0^∞ is a class of infinitely differentiable finite functions in two-dimensional Euclidean space, $C^*_{1/2}(\Omega)$ is the space of functions with norm

$$\|f\|_{C^*_{1/2}} = \max |f(x,y)\rho^{1/2}| \quad ((x,y) \in \Omega)$$

where ρ is the distance to the closest boundary point of the region $\bar{\Omega}$, $L_p(\Omega)$ ($p \geq 1$) is the space of functions integrable in Ω with degree p, and $\mathring{H}_\mu(\Omega)$, $H_\mu(\Omega)$ is the space of generalized Sobolev–Slobodetskii functions, where $H_0(\Omega) = L_2(\Omega)$.

Theorem 1 [13]. Suppose $g(x,y) \in C^2(\Omega)$ in the region Ω with boundary $\partial\Omega = \Gamma$: $F(X, Y) = 0$ ((X, Y) $\in \Gamma$), having continuously differentiable curvature. Then, integrable equation (2.1), (2.2), (3.1) (m = 1) is uniquely solvable in the space $C^*_{1/2}(\Omega)$ and the following correctness relation holds: $\|q\|_{C^*_{1/2}} \leq M\|g\|_{C^2}$ (M = const).

Since we have the equality $\rho = |F(x,y)|[F_{,x}^2(X, Y + F_{,y}^2(X, Y)]^{-1/2}$, the correct solvability in the class $C^*_{1/2}(\Omega)$ allows of the presence of a singularity in the function $q(x,y)$ of the form $|F(x,y)|^{-1/2}$ as $(x,y) \to (X, Y)$. Hence, we obtain the following structure of the solution of the equation

$$q(x,y) = \omega(x,y)|F(x,y)|^{-1/2} \tag{3.2}$$

where $\omega(x,y) \in C(\Omega)$ is a function that is continuous in the closed region $\bar{\Omega}$, i.e. $q(x,y) \in L_p(\Omega)$.

Theorem 2 [14]. Integral equation (2.1), (2.2), (3.1) ($m = 0$) is uniquely solvable in $C(\Omega)$ for any function $g(x,y) \in C(\Omega)$.

Theorem 3 [16]. If $q(x,y) \in H_\alpha(\Omega)$ ($\alpha \geq \beta = m/2$, $m = 2.4$) in the region Ω with smooth boundary $\partial\Omega = \Gamma$ ((X, Y) $\in \Gamma$), a solution of integral equation (2.1), (2.2), (3.1) ($m = 2, 4$) exists and is unique in the class $\mathring{H}_{-\beta}(\Omega)$, the function $q(x,y)$ can be represented in the form

$$q(x,y) = \psi(x,y) + \Psi(x,y), \quad \psi(x,y) \in H_{\alpha-m}(\Omega) \tag{3.3}$$

Note that the unique solvability of integral equations (2.1), (2.2), (3.1) for $m = 2$ or $m = 4$, respectively, in the spaces $\mathring{H}_{-1}(\Omega)$ and $\mathring{H}_{-2}(\Omega)$ denotes the following. Suppose $j = (j_1, j_2)$ is a multi-index, $|j| = j_1 + j_2$. We will denote by $\{Q(X, Y)\delta(\Gamma)\}^{(j)}$ the distribution with carrier in Γ, which acts on the test function $f \in C_0^\infty$ by the formula

$$(\{Q(x,y)\delta(\Gamma)\}^{(j)}, f) = (-1)^{|j|} \int_\Gamma Q(X,Y) f^{(j)}(X,Y) ds$$

where $Q(X, Y)$ is a smooth function on Γ. Then, by (3.3), we can write

$$\Psi(x,y) = \sum_{|i|=0}^{\beta-1} (-1)^{|i|} \{Q_i(X,Y)\delta(\Gamma)\}^{(i)} \tag{3.4}$$

Hence we conclude that an increase in the smoothness of the function $g(x,y)$ under the condition of Theorem 3 does not disturb the structure of (3.3) and (3.4), but may increase the degree of smoothness of the correction $\psi(x,y)$ in (3.3). For example, if $g(x,y) \in H_m(\Omega)$, then $\psi(x,y) \in L_2(\Omega)$.

Hence, it follows from Theorem 1 that the contact pressure $q(x,y)$ has the form (3.2), i.e. it contains a singularity of the type $|F(x,y)|^{-1/2}$, when the properties of the coatings are modelled: (1) by the equations of the theory of elasticity, (2) by Eqs (1.10) when there is weak adhesion between the coating and the foundation, and (3) by relations (1.8)–(1.10), or the equations of the Melan covering (1.11), or the first two formulae of (1.11) in combination with theorems of the Reissner–Timoshenko or Kirchhoff–Love type, if the coating is rigidly attached to the substrate.

By Theorem 2 the function $q(x,y)$ has finite values on the boundary of the contact region, if the properties of the coatings are described by Eqs (1.5)–(1.7) or are modelled by a set of Winkler springs.

By Theorem 3 the contact pressure $q(x,y)$ is made up of a distributed load and point forces or forces and moments acting on the boundary of the contact region, when the properties of the coating are described by the Reissner–Timoshenko and Kirchhoff–Love equations respectively and it is weakly attached to the foundation.

To solve integral equations (2.1), (2.2), (3.1) one can use a modification of the variational-difference method, the scheme of which and its justification are given in [13]. Here it is only necessary to note that for the first problem, in the case of a coating modelled by a Kirchhoff–Love or a Reissner–Timoshenko type plate, it is necessary to formulate the conditions for determining $Q_0(X, Y)$ and $Q_1(X, Y)$ [16].

This research was supported financially by the Russian Foundation for Basic Research and the International Association for Promoting Cooperation with Scientists from the Independent States of the Former Soviet Union (95-IN-RU-492).

REFERENCES

1. POPOV, G. Ya. and TOLKACHEV, V. M., The problem of contact between rigid bodies with thin-walled elements. *Izv. Akad. Nauk SSSR. MTT*, 1980, 4, 192–206.
2. VASIL'YEV, V. V., A discussion on the classical theory of plates. *Izv. Ross. Akad. Nauk. MTT*, 1995, 4, 140–150.
3. KOVALENKO, Ye. V., More-accurate equations of the deformation of elastic plates. *Prikl. Mekh.*, 1989, **25**, 10, 111–116.
4. PETRASHEN', G. I., The theory of vibration of thin plates. *Uch. Zap. Leningr. Gos. Univ. Ser. Mat. Nauk*, 1951, **149**, 172–249.
5. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of the Theory of Elasticity*. Nauka, Moscow, 1974.
6. KOVALENKO, Ye. V., Contact problems for bodies with coatings (formulations and methods of solution). *Izv. Akad. Nauk ArmSSR. Mekhanika*, 1988, **41**, 40–50.
7. ALEKSANDROV, V. M. and MKHITARYAN, S. M., *Contact Problems for Solids with Thin Coatings and Layers*. Nauka, Moscow, 1983.
8. GRIGOLYUK, E. I. and TOLKACHEV, V. M., *Contact Problems of the Theory of Plates and Shells*. Mashinostroyeniye, Moscow, 1983.
9. VLASOV, V. Z. and LEONT'YEV, N. N., *Beams, Plates and Shells on an Elastic Foundation*. Fizmatgiz, Moscow, 1960.
10. PETRISHIN, V. I., PRIVARNIKOV, A. K. and SHEVLYAKOV, Yu. A., The solution of problems for multilayer foundations. *Izv. Akad. Nauk SSSR. Mekhanika*, 1965, 2, 138–143.
11. ALEKSANDROV, V. M., BABESHKO, V. A., BELOKON', A. V., VOROVICH, I. I. and USTINOV, Yu. A., The contact problem for a ring layer of small thickness. *Inzh. Zh. MTT*, 1966, 1, 135–139.
12. KOVALENKO, Ye. V., Calculation of the wear at the coupling between an axle and a hub. *Izv. Akad. Nauk SSSR. MTT*, 1982, 6, 66–72.
13. BABESHKO, V. A., GLUSHKOV, Ye. V. and ZINCHENKO, Zh. F., *The Dynamics of Non-uniform Linearly Elastic Media*. Nauka, Moscow, 1989.
14. KANTOROVICH, L V. and AKILOV, G. P., *Functional Analysis*. Nauka, Moscow, 1977.
15. ESKIN, G. I., *Boundary-value Problems for Elliptic Pseudo-differential Equations*. Nauka, Moscow, 1973
16. RAMM, A. J., *Fields Estimation Theory*. Longman/Wiley, New York, 1990.

Translated by R.C.G.